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# AN INVERSE PROBLEM IN DYNAMIC PROGRAMMING AND AUTOMATIC CONTROL

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The RAND Corporation
SANTA MONICA • CALIFORNIA

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## PREFACE

Part of the Project RAND research program consists
of basic supporting studies in mathematics. The mathe-
matical research presented here deals with an inverse
problem in automatic control theory.

## SUMMARY

The authors investigate the inverse of a basic problem of automatic control theory, namely, being given an optimal policy and the equations and inequalities representing the constraints of the problem, they seek the criterion function or functions that lead to this policy; methods of solution are found in some simple cases.

## CONTENTS

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# AND AUTOMATIC CONTROL

## 1. INTRODUCTION

One of the basic problems of modern control theory
is that of minimizing a functional

$$(1.1) \qquad J(x,y) = \int_0^T g(x,y)dt,$$

where $x$ and $y$ are N-dimensional vector functions
subject to the relations

$$(1.2)(a) \quad x(0) = c,$$

$$(b) \quad \frac{dx}{dt} = h(x,y),$$

$$(c) \quad r_i(x,y) \le 0, \quad i = 1,2,\ldots,k.$$

Problems of this nature may be attacked by means of
classical variational techniques and extensions [1], [2],
or by means of dynamic programming. The latter emphasizes
the physical concept of feedback control, since it
attempts to determine directly the optimal policy
$v = v(c,T)$ as a solution of the equation

$$(1.3) \qquad f_T = \max_v [g(c,v) + (\text{grad } f,h)],$$

where $r_i(c,v) \le 0, \quad i = 1,2,\ldots,k,$ and

$$(1.4) \qquad f(c,T) = \min J(x,y).$$

In this Memorandum we initiate investigation of an inverse problem of some importance: "Given the optimal policy $v = v(c,T)$ and the descriptive equations of (1.2), what criterion function possesses this optimal policy?" The study of inverse problems forms a chapter in the calculus of variations, with emphasis laid on determining Lagrangians having a given family of curves as Euler arcs. (See [4], [5].)

In the simple case where no constraints are present and $h(x,y) \equiv y$, we obtain a simple linear partial differential equation which, in principle, determines $f$ and $g$. In some cases, this leads to an effective analytic solution.

## 2. SCALAR CASE

Consider the problem of minimizing

$$(2.1) \qquad J = \int_0^T g(u,u')dt,$$

with $u(0) = c$, where $u(t)$ is a scalar function. Writing $f(c,T)$ for the minimum value, we obtain in the usual fashion [3] the equation

$$(2.2) \qquad f_T = \min_v [g(c,v) + vf_c].$$

This in turn leads to the two equations

$$(2.3) \qquad f_T = g(c,v) + vf_c,$$

$$0 = g_v + f_c,$$

or

$$(2.4) \qquad f_T = g - vg_v,$$

$$f_c = -g_v.$$

Eliminating $f$ by partial differentiation, we have the equation

$$(2.5) \qquad (g - vg_v)_c = -(g_v)_T,$$

which, in principle, determines $g$, given the function $v(c,T)$.

This equation in $g$ involves second partial derivatives with respect to $v$ and is not immediately tractable.

## 3. THE FIRST-ORDER EQUATION FOR $f_T$

Returning to (2.3), let us proceed in the following fashion. Write

$$(3.1) \qquad (f_T - vf_c)_T = g(c,v)_T = g_v v_T = -v_T f_c,$$

upon taking account of the second equation of (2.3). Hence,

$$(3.2) \qquad f_{TT} - v_T f_c - vf_{cT} = -v_T f_c,$$

or

(3.3)     $(f_T)_T - v(f_T)_c = 0,$

a first-order linear equation for $f_T$, given the function $v = v(c,T)$. Once $f$ is determined, we obtain $g$ from (3.3).

As is immediately seen, the determination of $f_T$ is quite simple if we ask for a criterion function which yields an optimal policy of the form $v(c,T) = \phi(c)k(T)$.

## 4. MULTIDIMENSIONAL CASE

Consider the multidimensional problem. The basic equation is

(4.1)     $f_T = \min_{v} [g(c,v) + (\text{grad } f, v)].$

In terms of the components, we obtain the equations

(4.2)     $f_T = g(c,v) + \sum_{i=1}^{N} f_{c_i} v_i,$

$0 = g_{v_i} + f_{c_i},$

which yield, in exactly the same fashion as before, the equation

(4.3)     $(f_T)_T = \sum_{i=1}^{N} (f_T)_{c_i} v_i.$

Again, the determination of $f_T$, given that
$v(c,T) = \varphi(c)k(T)$ (where $k(T)$ is a scalar and $\varphi(c)$
is a vector), is reduced to a conventional problem in
first—order partial differential equations which can be
treated by means of characteristics or otherwise.

## 5. DISCUSSION

Finally, let us note that if one of the significant
functions of a variational problem is given, it is
frequently possible to form equations for the others in
terms of it. For example, if the integrand $g(c,v)$
is given, then it is possible to find a first—order
partial differential equation for the optimal control
function $v(c,T)$. In fact, upon expanding, we see that
equation (2.5) yields the desired relationship. These
observations have interesting consequences in analytical
dynamics and control theory which will be discussed
elsewhere.

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